SELF-CONTAINED PHASE-SPACE FORMULATION OF QUANTUM MECHANICS AS STATISTICS OF VIRTUAL PARTICLES

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НАЯ ФОРМУЛИРОВКА КВАНТОВОЙ МЕХАНИКИ КАК СТАТИСТИКА
ВИРТУАЛЬНЫХ ЧАСТИЦ.

Исходным положением настоящей работы является констатация, что конвен-
циональный квантово-механический формализм существенно неполон. Как допол-
нение этого формализма предложено определенное стохастическое множество вир-
туальных частиц. Подобное множество присутствует, хотя и неявным образом, как
в классической механике, так и в формулировке квантовой механики в фазовом
пространстве.

I. ON THE PHASE-SPACE FORMULATION OF QUANTUM MECHANICS. AIM AND ORGANIZATION OF THE PRESENT STUDY

The phase-space formulation of quantum mechanics has its origins in the work done by W i g n e r [1] and W e y l [2] . Basically, it amounts to the assertion that to each state function $\Psi(q, t)$ of a given quantum-mechanical "particle" of mass m and to each quantum-mechanical "observable" $\hat{A}(\hat{q}, \hat{p})$ correspond two functions: $F(q, t, p)$ and $A(q, p)$, defined on the classical phase space of a particle (material point) with mass m, such that the equality

$$(I.1) \qquad \langle \hat{A} \rangle \equiv \int \Psi^*(q, t)\hat{A}\Psi(q, t)dq = \int\int F(q, t, p)A(q, p)dqdp \equiv \langle A \rangle$$

and the marginal conditions

$$(I.2a) \qquad \int F(q, t, p)dp = \Psi^*(q, t)\Psi(q, t)$$

$$(I.2b) \qquad \int F(q, t, p)dq = \Psi^*(p, t)\Psi(p, t)$$

hold ($\Psi(p, t)$ is the state function in p-representation). The equality (I.1) obviously means that the basic quantum-mechanical quantity $\langle \hat{A} \rangle$ can be represented, at least formally, as the mean value $\langle A \rangle$ of some quantity in phase space.

Although this formulation is over half a century old, [3] it is difficult to determine its status in quantum mechanics. There exist contradictory views on this problem. There are authors, e.g. d e G r o o t and S u t t o r p, who believe [4] that this is an alternative formulation of quantum mechanics. Most authors, however, are far from the idea of equivalence between the phase-space formulation and the conventional formulation. A characteristic example may be the first sentence of a recently published fundamental survey on that problem [5] (one of whose authors is W i g n e r): "It is well known that the uncertainty principle makes the concept of phase space in quantum mechanics *problematic*" (italic is mine, A.A.). In fact, for a number of authors [6—9] the uncertainty (indeterminacy) principle serves as sufficient grounds for a categorical rejection of any application of the classical phase space in quantum mechanics " ... where the relation of indeterminacy is always at hand... so that not only the continuous phase space but also its discrete decomposition into cells are devoid of meaning" [6] .

The relations of uncertainty are not the sole occasion for doubts about the physical cogency of the phase-space formulation of quantum mechanics. Some aspects of this formulation, which probably make it seem incompetitive with respect to the conventional formulation, are the following:

1. The phase-space distributions are not determined uniquely by the state function: corresponding to each state there is an entire set of complex in general Wigner–Cohen [10] functions $F(q, t, p)$ which satisfy (I.2) and may be used for the representation (I.1).

2. Those of them which are bilinear functionals of Ψ are not always positive definite, [11] hence they are often called quasidistributions.

3. None of the Wigner–Cohen functions satisfies the Liouville equation. Consequently, the quasidistributions cannot be interpreted as densities of ensembles of classical particles. In such a case the question inevitably arises: Can the phase-space of a classical material point be the arena for statistics of entities different from the classical material points, entities which moreover allow for (and even require) a signed (i.e. non positive-definite) density? The negative answer to this question seems to be self-imposing: "Suppose now that $\mathfrak{S}$ is a system which does not follow the laws of classical mechanics. Then one cannot associate with it a phase-space in general." [12] We shall see, however, that this conclusion, which seems forgone, is actually illogical.

Another argument against the equivalence of the phase-space formulation with the conventional formulation is that the return from operators to ordinary phase-space functions (as "observables") looks like an unacceptable regression, all the more so because also in the phase-space formulation, in the final count, the phase-space functions $A(q, p)$ are determined through the operators (by the familiar correspondence rules [13, 14] and not vice versa. Indeed, in the known phase-space formulations a primary meaning of "observables" is attributed only to the operators, while the classical dynamic quantities $A(q, p)$ in (I,l) are considered as secondary objects defined through the "observables" and devoid of any independent physical meaning. Likewise, no independent meaning is attributed to the phase densities (densities — of what?) $F(q, t, p)$ and of the local quantities FA (the integrands in (I.1)). It seems that the most widespread belief about the phase-space formulation of quantum mechanics is that it is a formal approach which, in a number of cases, proves to be inexplicably useful and effective, although it could not possibly have an independent physical meaning.

This paper is an attempt to demonstrate that this is not so, despite the fact that the sceptic attitude toward the phase-space formulation is not unfounded. Actually, what are the grounds for that?

In my opinion the basic shortcoming of the phase-space formulation lies in its incompleteness. There is no answer to the most essential question: Distribution of what is the $F(q, t, p)$ distribution? We know that, as a matter of principle, conventional quantum mechanics declines to answer the question as to what actually is the quantum mechanical "particle" when no one looks at it. A lot of efforts have been expended to declare this question meaningless, but nevertheless this is hardly a solution. So density of what is $F(q, t, p)$? In classical statistical mechanics the non-negative phase density describes ensembles of well defined entities — classical particles. They **cannot** be the object of quantum mechanics. But then we return to the **fundamental** question:

Is it possible that entity exists, which is sufficiently different from the classical material point to have quantum mechanical behaviour and, at the same time, to be sufficiently kindred to the classical material point as to be placed in its phase space?

The conventional phase-space formulation, following the orthodox interpretation of quantum mechanics, has declined entirely to investigate this problem which, actually, is fundamental for any phase-space formulation. This refusal leads to the necessity to fill the gap caused by it, as in the case of the ordinary interpretation, with the artificial and created *ad hoc* operator theory of measurement.

The purpose of the present paper is to offer and substantiate a positive answer to the fundamental problem formulated above, and thereby to demonstrate that the conclusions, based on the principle of uncertainty, about the inapplicability of the concept of phase-space in quantum mechanics are actually groundless. The idea is that $F(q, t, p)$ is the distribution density of really existing though supershortliving ultramicroobjects ("particles-phantoms") which we shall call **quantons**. We shall see that, from a mathematical viewpoint, the quantons are in a certain sense trivial objects although physically they can, initially, cause confusion. Their introduction

makes it possible to overcome the objections resulting from the relations of uncertainty, as well as the difficulties of the phase-space formulation connected with the indefiniteness of the phase-space distributions. Besides, they make superfluous the specially *ad hoc* introduced operator theory of measurement and make it possible to shed new light on the role of the operators in quantum mechanics. Of course, all that is not obtained free of charge. The price paid is full renunciation (in a certain sense) of localness. It is quite possible that one may find this price too high, but this is a matter of taste.

In Section II we offer a possible interpretation of the non positive-definite phase-space distributions, which amounts to dualization of the dynamic characteristics (we introduce the concept of **anticharacteristics**). Besides, we make the transition to the seven-dimensions extended (with a temporal dimension) phase space. The quantons, with their dual antiquantons, are introduced in Section III. Their natural areal is exactly the extended phase space. We demonstrate that they have deep roots in both classical mechanics and in the conventional phase-space formulation of quantum mechanics. The connection between the quantons and the quantum-mechanical "particles" is postulated, and this makes it possible for the properties of these "particles" to be considered as properties of their quantons. The connection between the phase-space density of the quantons and the state vector is postulated in Section IV, and the customary operators are introduced (not postulated) on that basis. In the light of the above, Section V treats the reduction problem, while Section VI examines certain differences, observable in principle, between the quanton and the conventional approaches. Heisenberg's relations and the problem of the hidden parameters are the object of Section VII, while Section VIII draws a comparison between the conventional approach and the one offered in the present paper. The Conclusion, together with a general survey of the paper, treats also the problem of relativistic causality.

II. PHASE-SPACE DENSITY OF THE ELECTRON

The object to be examined, "particle in a quantum-mechanical sense", will be called briefly with the traditional term of electron (although in this paper we shall not take into account the existence of spin). It will become clear, from the definition to be given, that the electron differs (quite substantially) from the classical material point.

To describe the electron we shall use the quantities of mass m, charge e and the following well known spaces:

1. Ordinary three-dimensional (classical) physical space Q of the variables q (three-dimensional vectors), with (three-dimensional) volume element $dq \equiv dV$;

2. Three-dimensional moment-space P_m of the three-dimensional vectors of the moment p of a classical particle with mass m. The three-dimensional volume element of this space is dp;

3. One-dimensional time T with instants t and element time dt;

4. Four-dimensional classical (Galileo — Newtonian) spacetime QT of the variables q, t with (four-dimensional) volume element $dqdt$;

5. Six-dimensional phase-space QP_m of the variables q, p with volume element $dqdp$;

6. Seven-dimensional (extended) phase-space QTP of the variables q, t, p with volume element $dqdtdp$. Further on we shall use also the abbreviated notation

(II.1) $$(x) = (q, t, p), \; dx = dqdtdp.$$

The functions $A(x)$ defined in the QTP_m-space, which have physical meaning, will be called *physical characteristics* of the electron or simply *characteristics*. We have the *kinematic characteristics*

(II.2) $$q, t, v \equiv \frac{p}{m}$$

and the *dynamic characteristics*

(II.3)
$$A_1 = e, \; A_2 = m, \; A_3 = p, \; A_4 = \frac{p^2}{2m}, \; A_5 = eU(q, t), \; A_6 = A_4 + A_5, \; A_7 = q \times p,$$

each of them has well-known meaning ($q \times p$ is the vector product of the three-dimensional vectors q and p, i.e. the angular momentum).

A basic point in the present construction is the assumption that, parallel with the conventional dynamic characteristics, their respective dynamic anticharacteristics are also significant.

DEFINITION: *Dynamic anticharacteristic* corresponding to the dynamic characteristic $A(x)$ is the sign-conjugated to $A(x)$ QTP_m-function $(-A(x))$.

There is yet another function which will play a fundamental part in our examination. We shall introduce it by the following

POSTULATE: To each electron corresponds a QTP_m function $F(x)$ which satisfies the condition

(II.4) $$\int \int F(q, t, p)dqdp = 1.$$

This function shall be called *signed instant QP_m-density* of the electron. The term "signed" means that we do not presume $F(x)$ to be positively definite (negative values of $F(x)$ are admissible as well). The implications relative to the physical meaning and to the concrete choice of this function will be discussed in Section IV.

Indefiniteness of (a random) distribution appears to be a contradictory concept. To avoid this difficulty, we shall make use of the fact that to each function correspond naturally two *positive* functions

(II.5) $$F^+ = \frac{1}{2}(|F| + F)$$

(II.6) $$F^- = \frac{1}{2}(|F| - F)$$

which obviously satisfy the conditions

(II.7)
$$F^+ \geqq 0, \qquad F^- \geqq 0, \qquad F^+ F^- = 0,$$

(II.8)
$$F = F^+ - F^-,$$

(II.9)
$$|F| = F^+ + F^-.$$

Though non-normalized to unity, the functions $F^\pm$ will play an essential part in our investigation. In addition to these functions in the case of indefinite F it is exepedient to introduce also the function

(II.10)
$$\varepsilon(x) = \begin{cases} \frac{F(x)}{|F(x)|}, & \text{if } F \neq 0 \\ 0, & \text{if } F = 0 \end{cases}$$

(a "signed indicator"). Now we can write

(II.11)
$$F = |F|\varepsilon, \quad \varepsilon^2 = +1.$$

After the introduction of the anticharacteristics and the two functions $F^\pm$ the introduction of the following definitions is natural.

DEFINITION: The quantity

(II.12)
$$F^+ A = \begin{cases} FA, & \text{if } F \geqq 0 \\ 0, & \text{if } F \leqq 0 \end{cases}$$

is instant QP_m-density of the dynamic characteristic A of the electron; the quantity

(II.13)
$$F^-(-A) = \begin{cases} FA, & \text{if } F \leqq 0 \\ 0, & \text{if } F \geqq 0 \end{cases}$$

is instant QP_m-density of the dynamic anticharacteristic $(-A)$ of the electron (e.g. $F^-(-m)$ is instant QP_m-density of the *anti*mass of the electron).

In view of (II.8) we have

(II.14)
$$FA = F^+ A + F^-(-A),$$

i.e. FA may be viewed as a sum of densities of characteristics and anticharacteristics. As already remarked, the non-negative densities $F^\pm$ are not normalized toward unity (unlike the indefinite density F). This problem (actually the problem of "density/densities of what?") will become clear after the introduction of the quantons, and now we shall continue with the survey of the quantities inherent to the "indefinite formalism".

DEFINITION: The quantity

(II.15)
$$\int F^+ A\, dp$$

is the instant Q-density of the standard characteristic A of the electron; the quantity

(II.16)
$$\int F^-(-A)dp$$

is the instant Q-density of the anticharacteristic $(-A)$ of the electron.

DEFINITION: The sum of the last two densities

(II.17)
$$\int (F^+ - F^-)Adp \equiv \int FAdp$$

is the instant *effective Q-density* of the dynamic characteristic of the electron (the identity follows from (II.8)).

DEFINITION: The instant *effective integral value* of the dynamic characteristic A of the electron is the quantity

(II.18)
$$\int (F^+ - F^-)Adqdp \equiv \int FAdqdp.$$

The same quantity shall be called instant *integral value* of the dynamic characteristic A of the electron (without the adjective "effective").

The problem arises whether it would not be more expedient to go to the extended phase space QTP_m. This is certainly more natural since both the QP_m-density $F(x)$ and the QP_m-dynamic characteristics $A(x)$ are defined as functions precisely in the QTP_m-space. Moreover, in view of our purpose the transition from QP_m to QTP_m is not only natural and expedient, but necessary as well, since we will have to bring in a density defined directly in QTP_m. That is why we shall *postulate* the extension of $F(x)$ from QP_m into QTP_m.

The extensions of F and A from QP_m into QTP_m shall be introduced by means of the constant

(II.19)
$$N = \frac{mc^2}{h},$$

which obviously has the dimension of frequency and whose reciprocal value will be denoted by the symbol T_m

(II.20)
$$T_m = \frac{h}{mc^2} \longleftrightarrow NT_m = 1.$$

POSTULATE: The signed QTP_m-density of the electron is $NF(x)$.

Here N is defined by (II.19), $F(x)$ is the signed instant QP_m-density of the electron which setisfies (II.4).

DEFINITION: The quantity

(II.21)
$$\pm A(x)T_m$$

is the elementary action/antiaction of the characteristic A of the electron (e.g. the elementary action/antiaction of the quantity mc^2 is $\pm h$).

DEFINITION: The QTP_m-density of the elementary action/antiaction $\pm AT_m$ of the characteristic A of the electron is the quantity

$$(II.22) \qquad\qquad NF^{\pm}(\pm AT_m).$$

Corollary: In view of (II.20), (II.12) and (II.13), the QTP_m-density of the elementary action/antiaction of the characteristic A of the electron (II.22) prove to be identical with the instant QP_m-density (II.14) of the dynamic characteristic/anticharacteristic of the electron.

In other words, with the transition $QP_m \to QTP_m$ thus postulated, the product FA remains invariant

$$(II.23) \qquad\qquad FA \to (NF)(AT_m) = FA.$$

Corollary: The instant QP_m integral value of the dynamic characteristic of the electron (II.18) may be represented also by the QTP_m quantities

$$(II.24) \qquad\qquad \int FA\,dq\,dp = \int (NF)(AT_m)\,dq\,dp.$$

Concerning the condition for the normalization of the QTP_m-density NF of the electr   by virtue of (II.4) the following equation is valid in QTP_m

$$(II.25) \qquad\qquad \int_{t-T_m/2}^{t+T_m/2} NF(q, t, p)\,dq\,dt\,dp = 1,$$

i.e. NF is normalized to unity in each region of QTP_m with boundaries the two parallel six-dimensional planes $t \mp T_m/2 = $ const.

Finally, it should be pointed out that to the extended QTP_m-density NF there correspond naturally the two positive QTP_m-densities NF^+ and NF^-.

It will become clear in the following Section, after the introduction of the quantons, that the extension performed is not only formally possible but is necessary as well.

III. QUANTONS AND ELECTRONS

Let x' be a given point of the QTP_m-space, $\delta(x, x')$ — a Dirac delta function in QTP_m with support x', and ε' — one of the two numbers ± 1.

DEFINITION: The generalized function

$$(III.1) \qquad\qquad \varepsilon' \delta(x, x')$$

is a *quanton* given in QTP_m. The support $(x') = (q', t', p')$ of the delta function (III.1) is support also of the quanton, and t' is instant-support of the quanton.

At $\varepsilon' = \pm 1$ the quanton is positive/negative (briefly p/n-quanton). The negative quantons will be called *antiquantons*.

Origin of the quantons

The quantons are obviously potentially present in each function $F(x) \equiv |F(x)|\varepsilon(x)$, which may be represented as follows

$$(III.2) \qquad F(x) = \int F(x')\delta(x, x')dx' \equiv \int |F(x')|\varepsilon(x')\delta(x, x')dx'.$$

In this manner it is possible to represent not only each Wigner–Cohen function but also the phase-space QP_m-density ρ_0^6 of the classical material point. This problem is essential for the phase-face interpretation of quantum mechanics and that is why we shall dwell on it albeit briefly. As we know, the material point may be represented in QP_m by the density

$$(III.3) \qquad \rho_0^6(q, t, p) = \delta(q, q_H(t))\delta(p, p_H(t)).$$

Here $q = q_H(t)$ and $p = p_H(t)$ are the equations of the Hamilton QP_m-trajectory of the material point (assumed as given). We may now write

$$(III.4) \qquad \rho_0^6(x) = \int \rho_0^6(x')\delta(x, x')dx'$$

and this is the representation referred to. If the equations of the trajectory

$$(III.5) \qquad\qquad\qquad q = q_H(t')$$
$$(III.6) \qquad\qquad\qquad p = p_H(t')$$
$$(III.7) \qquad\qquad\qquad t = t_H(t') \equiv t'$$

are written compactly in the following manner:

$$(III.8) \qquad\qquad\qquad x = x_H(t')$$

then the density (III.3) may be represented also as

$$(III.9) \qquad\qquad \rho_0^6(x) = \int \delta(x, x_H(t'))dt'.$$

According to the definition given above, the integrand is nothing but a positive quanton with point $x_H(t')$ as support. It follows that the quantons are connected with $\rho_0^6(x)$ not only by (III.4) but also directly through (III.9).

Our **basic idea** can now be formulated in the following two theses:

1. The extended phase-space density (the QTP_m-density) $NF(x)$ of the electron

$$(III.10) \qquad NF(x) = \int \rho(x')\varepsilon(x')\delta(x, x')dx',$$

not only can be potentially presented as average density of quantons, but also it is actually a (ensemble) density of quantons. In brief, the quantons actually exist.

2. The set of the quantons associated with a given electron is discrete (not continual).

Of course, discreteness (quantization) is a basic principle in quantum physics. We proceed to the description of a discrete set of quantons which is to play a basic part in our approach.

Let $\{\overset{n}{x}\}$ be a given discrete (finite or enumerable) set of points (lattice) in QTP_m (with set of the indices $\{n\}$) and let $\varepsilon(x)$ be a given signed indicator in QTP_m.

DEFINITION: The signed ultramicro QTP_m-density

$$(III.11) \qquad \Gamma(x) = \sum_n \varepsilon(\overset{n}{x})\delta(x, \overset{n}{x}),$$

where $n \in \{n\}$, will be called *generation of quantons* $\Gamma(x)$ with *support* the lattice $\{\overset{n}{x}\}$, provided the arrangement in $\{\overset{n}{x}\}$ is in correlation with the time, i.e. provided that

$$(III.12) \qquad \overset{m}{t} < \overset{n}{t} \rightarrow m < n$$

(m and n are numbers of quantons, while $\overset{m}{t}$ and $\overset{n}{t}$ are the respective instant-supports).

DEFINITION: A generation of quantons $\Gamma(x)$ is stochastic, when the points $\{\overset{n}{x}\}$ of its lattice are distributed in the QTP_m with a certain probability density. Under "generation of quantons" we shall always mean "stochastic generation of quantons". Besides that, we always assume that each generation of quantons is an element of a statistical ensemble of generations of quantons. A statistical ensemble of generations of quantons is each potentially unlimited set of generations which have one and the same probability QTP_m-density.

In our subsequent presentation, instead of "distribution of the supports of the quantons (from a given generation)" we shall refer more briefly to "distribution of the quantons (from a given generation)".

We now have at our disposal everything necessary to state our

BASIC POSTULATE: To each electron correspond:

I. One signed QTP_m-density $NF(x)$.

II. One generation of quantons $\Gamma(x) = \sum_n \varepsilon(\overset{n}{x}) \delta(x, \overset{n}{x})$ with QTP_m-density of the p/n-quantons NF^+ ($F(x)$ satisfies (II.25), N is determined from (II.19), $\varepsilon(x)$ from (II.10), and F^+ from (II.5,6)). In brief, an electron is a pair $(F(x), \Gamma(x))$.

Instead of "the quantons from the generation of quantons corresponding to a given electron" we shall further refer briefly to "the quantons of a given electron".

According to the last postulate, the QTP_m-density of the supports of the quantons of a given electron is $N|F(x)|$.

DEFINITION: The marginal densities

$$(III.13) \qquad \int NF^{\pm}(q, t, p)dp$$

are the QT-densities of the p/n-quantons of a given electron.

DEFINITION: The difference of QT-densities of the quantons and antiquantons

$$(III.14) \qquad \int N(F^+ - F^-)dp = NW(q, t)$$

is the Q-density of the *effective* quantons of a given electron.

Corrolary: It follows from (II.8) and (II.4) that the quantity $W(q, t)$ (introduced by the last equation) satisfies the conditions

$$(III.15a) \qquad W(q, t) = \int F(q, t, p)dp,$$

$$(III.15b) \qquad \int W(q, t)dq = 1.$$

Corresponding to each quanton with support $(\overset{n}{x}) = (\overset{n}{q}, \overset{n}{t}, \overset{n}{p})$ is an event $(\overset{n}{q}, \overset{n}{t})$ in space-time QT. This event will be called *the appearance-disapperance* in the point $\overset{n}{q}$ at the moment $\overset{n}{t}$ of the given quanton. Thus the density NW determined from (III.14) is a density of events in the space-time QT. However, to each density $\rho(q, t)$ of events in space-time corresponds naturally the quantity Q-density of a *frequency* of events taking place in the Q-space. Indeed, for a given QT-density $\rho(q, t)$ of events we have the following formula for the number of events dn which lie in the $dqdt$ element

$$(III.16) \qquad dn = \rho dq dt.$$

Hence $(dn/dt) = \rho dq$ is the frequency $d\nu$ with which these events take place in the dq volume, i.e.

$$(III.17) \qquad d\nu = \rho dq.$$

Finally, $\rho = d\nu/dq$ is the Q-density of this frequency. It follows that the frequency $d\nu$ with which the effective quantons appear and disappear in the dq volume (in this case $\rho = NW$) is

$$\text{(III.18)} \qquad d\nu = NW\,dq.$$

Using the above and (III.15), we obtain that the integral frequency $\int\limits_{\infty} d\nu$ with which the effective quantons appear and disappear in the entire Q-space is $N = mc^2/h$. In other words, each $N^{-1} = T_m = h/mc^2$ seconds, somewhere in space probably appears and disappears an effective quanton with density of the probability $W(q, t)$.

Equation (III.18) plays a major part in the interpretation offered and we shall return to it in Section V. For the time being we shall go on with the introduction of new concepts connected with the discretization of the set of quantons.

DEFINITION: Ultramicro QTP_m-density of the elementary action AT_m of the dynamic quantity A of the electron is the QTP_m-function

$$\text{(III.19)} \qquad \sum_n \varepsilon(\overset{x}{x})\delta(x, \overset{n}{x})A(x)T_m,$$

(where $\sum\limits_n \varepsilon(\overset{n}{x})\delta(x, \overset{n}{x})$ is the generation of quantons which corresponds to the given electron).

In the following definition Δx is a physically infinitely small QTP_m-volume in which the functions $A(x)$ and $\varepsilon(x)$ may be considered as constant.

DEFINITION: *Micro QTP_m-density* of the elementary action AT_m of the dynamic quantity A of the electron is the average value

$$\text{(III.20)} \qquad \frac{1}{\Delta x}\int\limits_{\Delta x}\sum_n \delta(x, \overset{n}{x})\varepsilon(x)AT_m\,dx.$$

Since, according to the condition given, $A(x)$ and $\varepsilon(x)$ may be considered constant in Δx, it follows that the density (III.20) may be represented also in the following manner

$$\text{(III.21)} \qquad \frac{1}{\Delta x}\varepsilon(x)A(x)T_m \int\limits_{\Delta x}\sum_n \delta(x, \overset{n}{x})\,dx$$

(where x is an arbitrary point from Δx). However, the integral here is equal simply to the probable number $N|F|\Delta x$ of the supports $\overset{n'}{x}$ which fall into the element Δx, and since $\varepsilon|F| = F$ and $NT_m = 1$ it turns out in the final count that (III.20) in the quantity $FA \equiv |F|\varepsilon A$ studied in the preceding Section. So if we assume the quantons to be real physical objects, FA acquires direct physical meaning as average instant QP_m-microdensity of the dynamic characteristic *A of the quantons*

of the given electron. After we have shown that (III.20) coincides with FA, the quantity (II.17) (i.e. the instant effective Q-density of the dynamic characteristic A) gets a "quanton" interpretation as an instant Q-microdensity of the dynamic characteristic A of the *effective quantons* of the electron. Likewise, the quantity $\int FAdqdp$ (II.18), introduced above as instant effective integral value of the dynamic characteristic A of the electron, obtains direct quanton interpretation as instant QP_m-integral value of the averaged dynamic characteristic A of the effective quantons. In each phase-space formulation the quantity $\int FAdqdp$ is identified with the so-called quantium-mechanical "observable" mean value, but, as a matter of principle, **no** physical meaning is attributed to the integrand. It follows from what has been said so far that in the proposed theory the quantons play the fundamental role of real albeit supershortliving carriers of all dynamic characteristics of the electron. In this context we shall discuss briefly the problem of the dynamic characteristics of the individual quantons. In analogy with (II.24) the following definition makes sense.

DEFINITION: The instant integral dynamic characteristic $\overset{n}{X}_i$ of an individual quanton corresponding to A_i (see (II.3)) is the quantity

$$(\text{III.22}) \qquad \overset{n}{X}_i(t) = \int \delta(q, t, p; \overset{n}{q}, \overset{n}{t}, \overset{n}{p})\varepsilon(q, t, p)A_i(q, t, p)T_m dqdp.$$

Corrolary:

$$(\text{III.23}) \qquad \overset{n}{X}_i(t) = \varepsilon(\overset{n}{x})\delta(t, \overset{n}{t})A_i(\overset{n}{x})T_m.$$

Corrolary: The instant integral mass $\overset{n}{M}$ of the individual quanton is the quantity

$$(\text{III.24}) \qquad \overset{n}{M}(t) = \varepsilon(\overset{n}{x})\delta(t, \overset{n}{t})\frac{h}{c^2}$$

(follows from $A_i = m$ and (II.20)).

Corrolary: The individual p/n-quanton may be interpreted as a virtual (supershortliving) classical particle with infinitely great (in absolute value) positive/negative mass

$$(\text{III.25}) \qquad \overset{n}{M} = \varepsilon(\overset{n}{x})\frac{h}{c^2}\delta(t, \overset{n}{t}).$$

This "particle-phantom" appears at a moment $\overset{n}{t}$ in a point $\overset{n}{q}$, it exists during an infinitely short time $\delta\overset{n}{t} = (\delta(t, \overset{n}{t}))^{-1}$ and disappears. The infinitely great mass and the infinitely short lifetime of the quanton have a well determined product:

$$(\text{III.26}) \qquad \varepsilon(\overset{n}{x})\overset{n}{M}c^2\delta\overset{n}{t} = h,$$

where the action quantization is directly expressed.

Corrolary: The seven linear by mass instant integral dynamic characteristics of the individual quanton may be represented in the following manner

$$\overset{n}{X_i} = \frac{\overset{n}{M}}{m} A_i \equiv \varepsilon(\overset{n}{x})\delta(t, \overset{n}{t})A_i(\overset{n}{x})T_m.$$

(III.27)

For instance, for the momentum of the quanton we have $\overset{n}{P} = \overset{n}{M}\overset{n}{v}$ where $\overset{n}{v} = \overset{n}{p}/m$ (according to (II.2)) is the velocity of the quanton (regardless of whether it is positive of negative). The kinetic energy of the quanton is $\overset{n}{M}\overset{n}{v^2}/2$, etc.

Corrolary: The averaged in time value of the characteristic (III.27) for time interval T_m (which includes the moment $\overset{n}{t}$) is $\varepsilon(\overset{n}{x})A_i(\overset{n}{x})$. For instance, the mass of a p/n-quanton averaged for time T_m is $\pm m$ (for the momentum, kinetic energy, etc., we have $\pm\overset{n}{p}$, $\pm m\overset{n}{v^2}/2$ etc.).

Hence the quantons are sufficiently kindred with the classical material point so as to be contained in its (extended) phase space, while at the same time they differ (essentially) from it because they are not bound in a continuous trajectory but constitute a discrete lattice with stochastic distribution.

IV. DETERMINATION OF THE PHASE-SPACE DENSITY OF THE ELECTRON THROUGH THE STATE VECTOR. INTRODUCTION OF THE OPERATORS

The physical meaning of the signed phase-space density $F(q, t, p)$, according to the basic postulate from Section III, is the following: $F^{\pm}dqdp$ is the number dn of the p/n-quantons which, for a time $dt = h/mc^2$, appear and disappear in the phase volume $dqdp$. We must now treat the problem of the choice of the function $F(q, t, p)$. Its definition will be made by means of the state vector. In our further presentation

a. We shall presume as familiar the meaning of the concept of state space, as well as of the symbols $|\Psi\rangle$, $|q\rangle$, $|p\rangle$, $\langle\Psi,|$ etc. (the Dirac bracket notation) without assuming the validity of Schrödinger's equation;

b. $\hat{I}$ shall denote the unit operator, while the asterisk* shall denote complex conjugation;

c. The absence of h in a formula will mean that this formula has been written in a system in which $h = 1 = c$ (accordingly $\hbar = (2\pi)^{-1}$). For completeness we adduce the relations

(IV.1)
$$\int |q\rangle dq\langle q| = \hat{I} = \int |p\rangle dp\langle p|,$$

(IV.2)
$$\langle q'|q\rangle = \delta(q, q'), \quad \langle p'|p\rangle = \delta(p, p'),$$

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$$(\text{IV}.3a) \qquad \langle q|p\rangle = e^{i2\pi p \cdot q} = (\langle p|q\rangle)^*,$$

$$(\text{IV}.3b) \qquad \langle q|\Psi\rangle = \Psi(q, t), \quad \langle p|\Psi\rangle = \Psi(p, t).$$

As we know, to each electron corresponds a (normalized) state vector $|\Psi\rangle$ belonging to the linear vector (Hilbert) states space. Traditionally, the geometrical language is used only thus far and the tensor $|\Psi\rangle\langle\Psi|$ is called matrix (or operator) of the density. It would be more natural for $|\Psi\rangle\langle\Psi|$ to be called *state tensor* and then, together with the *state tensor* and the *state vector*, it would be quite natural to devote some attention also to the two (complex conjugated) *state scalars*

$$\langle\Psi|q\rangle\langle q|p\rangle\langle p|\Psi\rangle \qquad \text{and} \qquad \langle\Psi|p\rangle\langle p|q\rangle\langle q|\Psi\rangle$$

(i.e. the scalar products of the components $|q\rangle\langle q|\Psi\rangle$ and $|p\rangle\langle p|\Psi\rangle$) as well as of their mean value which coincides with the real part of each one (e.g. the first) of them

$$\frac{1}{2}\langle\Psi|(|q\rangle\langle q|p\rangle\langle p| + |p\rangle\langle p|q\rangle\langle q|)|\Psi\rangle \equiv \operatorname{Re}(\langle\Psi|q\rangle\langle q|p\rangle\langle p|\Psi\rangle).$$

The identification of the signed phase-space density of the electron with that real (but not positive in general) scalar will be accepted as

BASIC POSTULATE:

$$(\text{IV}.4a) \qquad F(q, t, p) = \operatorname{Re}(\langle\Psi|q\rangle\langle q|p\rangle\langle p|\Psi\rangle).$$

In a coordinate representation (IV.4a) reads

$$(\text{IV}.4b) \qquad F(q, t, p) = \operatorname{Re}\big(\Psi^*(q, t)e^{i2\pi p \cdot q}\Psi(p, t)\big).$$

This function (belonging to the Wigner–Cohen class of functions) has been introduced by T e r l e t s k y [15] and studied by M a r g e n a u and H i l l [16], M e h t a [17] and other authors. Further we shall use also the complexified function of Terletsky, which will be denoted by $\tilde{F}$:

$$(\text{IV}.4c) \qquad \tilde{F}(q, t, p) = \Psi^*(q, t)e^{i2\pi p \cdot q}\Psi(p, t).$$

Corrolary: Following directly from (IV.4) is the validity of the marginal conditions (I.2) which are satisfied also by the complexified density (IV.4c).

Let us make the following methodological remark: in a phase-space formulation the most fundamental part is that of the phase-space density. Consequently, its postulation may be substantiated only by methodological considerations (of simplicity, etc.).

INTRODUCTION OF OPERATORS

Let $A(q, t, p)$ be a phase-space function (with or without physical meaning). In view of (IV.4a) the density FA may be represented as follows

(IV.5)
$$FA = \text{Re}(\langle\Psi|q\rangle\langle q|A(q, t, p)|p\rangle\langle p|\Psi\rangle).$$

This reading suggests that FA may be written also by means of a linear operator $\hat{A}(t)$ in the following manner

(IV.6)
$$FA = \text{Re}(\langle\Psi|q\rangle\langle q|\hat{A}(t)|p\rangle\langle p|\Psi\rangle).$$

A sufficient condition for this representation is for the operator $\hat{A}$ to be defined as the solution of the operator equation

(IV.7)
$$\langle q|\hat{A}|p\rangle = \langle q|A(q, t, p)|p\rangle,$$

and this solution is the following

(IV.8)
$$\hat{A} = \int\int |q\rangle dq\langle q|A(q, t, p)|p\rangle dp\langle p|.$$

By means of the last equality, a linear (pseudodifferential) operator is juxtaposed uniquely to each phase-space function (for which the integral (IV.8) is meaningful). Conversely, through (IV.7) viewed as equation for $A(q, t, p)$, to each operator $\hat{A}(t)$ is juxtaposed uniquely the phase-space function

(IV.9)
$$A(q, t, p) = \frac{\langle q|\hat{A}(t)|p\rangle}{\langle q|p\rangle} \equiv \langle p|q\rangle\langle q|\hat{A}(t)|p\rangle \equiv \langle q|\hat{A}(t)|p\rangle\langle p|q\rangle.$$

In this manner, on the basis of the phase-space density (IV.4), a one-to-one correspondence (correspondence rule, association rule) is established between the phase-space functions $A(q, t, p)$ and the linear operators $\hat{A}$ operating in the linear space of the state vectors (actually $\hat{A}(t)$ is a second-rank tensor with components $A(q, t, p)$ in the mixed tensor basis $|q\rangle dq\langle q|p\rangle dp\langle p|$). In this sense the operators do not have the statute of entities with fundamental physical meaning which is possessed by the *dynamic characteristics*, the *phase-space density* and the *quantons* themselves.

Corolaries: From (IV.6) and (IV.1) we get the equalities

(IV.10)
$$\int FAdp = \text{Re}(\langle\Psi|q\rangle\langle q|\hat{A}|\Psi\rangle) \equiv \text{Re}(\Psi^*(q, t)\hat{A}_q(t)\Psi(q, t)).$$

(IV.11)
$$\int FAdpdq = \text{Re}(\langle\Psi|\hat{A}|\Psi\rangle) \equiv \text{Re}\int\Psi^*(q, t)\hat{A}_q(t)\Psi(q, t)dq$$

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($\hat{A}_q$ denotes the operator $\hat{A}$ in the q-representation).

The operator Re in (IV.11) becomes superfluous when the operator $\hat{A}$ (defined from (IV.8)) is a Hermitian one. In this case (IV.11) becomes the initial equation (I.1).

Corrolary. The operators corresponding according to (IV.8) to the seven basic dynamic characteristics A_i (II.3) are exactly the well-known standard quantum mechanical Hermitian operators.

It must be pointed out, however, that reality of the function $A(q, t, p)$ in (IV.8) **does not** ensure hermiticity of $\hat{A}(t)$. This cannot be an objection (of a logical nature) against the consecutive phase-space approach proposed, since it is **not** based on the operator theory of observation and since, in particular, this approach does not call for hermicity of the operators. Non-conformity with the conventional approach is at hand, for instance, in the case of the operator of the radial momentum, and also in the case of the operator of the square of the angular momentum. These discrepancies are essential on a conceptual plane and we shall examine that in Section VI. Now we shall discuss the dynamics of the phase-space density F.

Of course, the dynamics of F is determined uniquely by the dynamics of Ψ, i.e. the Schrödinger's equation must lead to an equation for F. Indeed, if Ψ satisfies Schrödinger's equation, then $\tilde{F}$ satisfies the equation

$$(IV.12) \qquad \frac{\partial \tilde{F}}{\partial t} = \sum_{n=1}^{\infty} (\frac{\hbar}{i})^{n-1} \frac{1}{n!} \left[\frac{\partial^n H}{\partial q^n} \frac{\partial^n \tilde{F}}{\partial p^n} - \frac{\partial^n H}{\partial p^n} \frac{\partial^n \tilde{F}}{\partial q^n} \right] .$$

which at $h = 0$ is obviously reduced to the classical Liouville equation. Conversely, (IV.12) and (IV.4) imply the validity of Schrödinger's equation. So in a consecutive phase-space approach, instead of Schrödinger's equation it is possible to postulate the phase-space equation (IV.12) which is a special case of the equation given by C o h e n in [10] as (5.1). As for the equation (IV.12), cf. [18–20].

However, there is another way of phase-space working out of Schrödinger's equation which, to the best of the author's knowledge, has not appeared anywhere. This is based not on the postulation of (IV.12), but on the postulation of the condition

$$(IV.13) \qquad \int \tilde{F}.(H + \frac{\partial \tilde{S}}{dt}) dp = 0,$$

where $\tilde{F}$ is the complexified Terletsky function (IV.4c), $H = H(q, t, p = \frac{\partial \tilde{S}}{\partial q})$ is the classical Hamiltonian, while the complex function

$$(IV.14) \qquad \tilde{S} = \tilde{S}(q, t, p) = \Phi_\Psi(p, t) + p.q + \frac{\ln A_\Psi(p, t)}{i2\pi}$$

is defined by means of the real phase $\Phi_\Psi(p, t)$ and amplitude $A_\Psi(p, t)$ of the wave function in p-representation: $\Psi(p, t) = A_\Psi(p, t) \times \exp(i2\pi\Phi_\Psi(p, t))$. It is not

difficult to see that under these conditions Schrödinger's non-relativistic equation is a direct corollary of (IV.13). The converse is also true. It should be pointed out that the trivial case of (IV.13) obviously is reduced to the Hamilton–Jacobi equation.

We already have what is necessary to treat the problem of the experimental determination of the physical characteristics of the electron.

V. QUANTONS, THE OBSERVABLE "POSITION", REDUCTION AND THE PRINCIPLE OF SUFFICIENT CAUSE

In this section we shall study the problem of the localization (the reduction of the wave function) of the electron in the light of the hypothesis of the existence of quantons. To that end we shall avail ourselves of the concept of IDEAL REGIS-TERING DEVICE (IRD) which will be characterized with volume ΔV and with the following

POSTULATE: A necessary and sufficient condition to register (localize) an electron in the volume ΔV of a given IRD is for one *effective* quanton of this electron to appear in the volume ΔV of the IRD.

It follows that the apperance of the first effective quanton of an electron in a given IRD leads to recording (localization) of the electron in the IRD. Clearly, in such a case the following is valid.

Corrolary 1: The probability of localization (registration) of the electron in the volume ΔV is proportional to the frequency with which its effective quantons appear in the volume ΔV.

Consequently, it is necessary to return to the problem of the frequency $d\nu$ with which the effective quantons appear in tne volume dq. It was established that this frequency is determined from formula (III.18); the quantity W participating in it is determined from (III.14), and after the introduction of the postulate (IV.4) we have for that frequency

$$(V.1) \qquad d\nu = \frac{mc^2}{h}\Psi^*(q,\,t)\Psi(q,\,t)dq.$$

We can now formulate yet another

Corollary 2: The probability of localizing the electron in the volume dq is

$$(V.2) \qquad W(q,\,t)dq = \Psi(q,\,t)\Psi^*(q,\,t)dq$$

(it follows from Corollary 1 and (V.1)).

In this manner, in the approach proposed, the *probability* of localizing the electron is related directly to the *frequency* of appearance of the effective quantons, whereas Born's postulate (V.2) turns out to be a corollary. It is necessary to remark yet another assertion which follows from the hypothesis of existense of quantons and is not only unknown in the conventional interpretation but clearly contradicts the assumption that it is complete. This is the assertion that there exists a (statistical) lower bound of the time necessary for the actuation of the registering device, this

lower bound being due not to characteristics of the device, but to the electron itself. This is the time

$$(V.3) \qquad \Delta t = \left(\frac{mc^2}{h} \int\limits_{\Delta V} \Psi^*(q,\,t)\Psi(q,\,t)dq \right)^{-1}$$

which, according to (V.1), is necessary for the appearance of an effective quanton in the volume ΔV of the registering device. The existence of such capture time is, as a matter of principle, subject to experimental verification.

The introduction of the quantons makes it possible to offer a natural model of the phenomenon called "reduction of the state" or, in the terminology adopted by von Neumann, "Process 1". According to von Neumann, whose opinion on this matter may be considered as accepted by all who adhere to the conventional interpretation, there are two fundamentally different types of interventions to which a system may be subjected. First, these are "the discontinuous, noncausal and instantaneously acting experiments or measurements ... (processes 1)". Second, this is a "type intervention in material systems ... given by the time-dependent Schrödinger differential equation which determines how the system changes continuously and causally in the course of time (process 2)" [21], Ch.V.1.

Unlike the conventional approach, in the one offered, the introduction of the quantons makes groundless the idea that the processes of type 1 shall be considered *acausal* and *instantaneous*. Besides, it becomes clear that the difference between the processes of types 1 and 2 is far from being fundamental.

In reality, trivially true (for such possible interpretation) is that the assertion "at the moment $t = 0$ the electron is in the registering device" is equivalent to the assertion: "at the moment $t = 0$, $\Psi = 0$ outside the registering device and $\Psi \neq 0$ in it". Let, at $t < 0$, $\Psi \neq 0$ in a *volume* W which contains (and possible exceeds many times) the volume ΔV of the registering device. Consequently, at $t < 0$ the boundary conditions for Ψ have been $\Psi = 0$ on the boundary of W, at the moment $t = 0$ these conditions have passed *discontinuously*, with a *jump*, on the surface of the registering device.

There are no obstacles to accept, however, that Ψ is subordinate to Schrödinger's equation all the time, including at the moment $t = 0$. In such a case the difference between processes 1 and 2 will turn out to be difference only in the type of the boundary conditions for Ψ — static (or quasistatic) for processes 2 and discontinuous in case 1 (c.f. the performance on a string instrument of the violin or, better still, of the guitar type). This obviously is not a difference of principle. The question is what could have caused the jump in the boundary conditions. The answer is: nothing could have caused this jump, according to the conventional interpretation in which, *by definition*, the description is *complete* only through Ψ, and it becomes necessary to assume that it is *acausal*: "nature... has disregarded the 'principle of sufficient cause'. [21] With the existence of the quantons it is absolutely natural to assume that the triggering of the registering device **is caused** by a quanton that has appeared in it: In this case causality becomes obvious,

reduction proves to be subordinate to the principle of sufficient cause, while Born's postulate about the meaning of $\Psi^*\Psi$ turns out to be a corollary of the postulate formulated at the beginning of this Section.

One possible answer to the question as to why precisely the effective quantons are playing a part in the reduction is the following: the registering device, which is a physical system in unstable energy equilibrium, can be triggered only by the introduction of positive energy (action, respectively) into it. An antiquanton introducing *negative* energy (action, respectively) not only does not destroy the equilibrium, but it even *stabilizes* it, destroying in advance the action of the first positive quanton that had appeared after it. As a result, it is precisely the frequency of appearance of the *effective* quantons that is significant to the actuation of the registering device.

VI. QUANTONS AND "OBSERVABLES"

As we have already stated, the operators of the seven dynamic characteristics (II.3), determined by the correspondence rule (IV.8), coincide with the conventional ones. Consequently, the average values of the quantum-mechanical "observables" — charge, mass. moment, kinetic, potential and full energy and angular momentum — coincide with the quantum averages of the respective quantities. The interpretation, however, is quite different. It is not at all necessary in the approach offered to postulate that to each "observable" corresponds an instrument which measures its values. On the other hand, with the existence of the quantons, it becomes a trivially explainable fact that the sole "observables" which are really observed (and not calculated) are the volume ΔV of the recording device and the instant t_0 (more accurately the time interval Δt_0) of localization. All other "observables" are in practice "definables" whose values are determined by means of the theory and of the real observables ΔV and Δt_0. Of course, this does not mean that the approach offered includes some theoretical ban on the direct observation of the local quantities $F^{\pm}A$ and the integrals from them. The possibility for direct observation of these quantities remains open for the time being.

In the approach offered the eigen-functions of the operators "observables" correspond to states in which the respective dynamic characteristic of the (quantons of) the electron has a constant marginal Q-density (no dispersion in co-ordinate space). However, this does not mean that, in the respective cases, there is no dispersion in the phase-space.

Concerning the "numerical" incongruences between the approach offered and the conventional one, they are due to the fact that for each phase-space formulation of quantum mechanics (i.e. for each choice of the Wigner — Cohen density) the correspondence

(VI.1) $$H(A(q,\,p)) \rightleftarrows H(\hat{A}(\hat{q},\,\hat{p}))$$

($H(x)$ is any function of x), which is considered as a basic one in the conventional interpretation, is violated [10] (in [10] this fact is considered as the reason why

true phase-space distribution cannot be defined at all). In particular, under the correspondence rule (IV.8) adopted in the present paper, corresponding to the phase-space function $\sum_n C_n(q, t)p^n$ is the operator $\sum_n C_n(\hat{q}, t)(\hat{p})^n$, i.e. we have

$$(VI.2) \qquad \sum_n C_n(q, t)p^n \rightleftarrows \sum_n C_n(\hat{q}, t)(\hat{p})^n$$

$(C_n(q, t))$ is a one-parameter family of QT-functions with parameter n (the same symbol is the exponent of the monomials p^n and $(\hat{p})^n$; $\hat{q}$ and $\hat{p}$ have the conventional meanings.) The proof of this last assertion is obvious, and almost obvious is the fact that at (VI.2) the correspondence (VI.1) is violated. A typical example of such a violation is the square of the angular momentum. Whereas the operator of the angular momentum determined by (VI.2) or (which is just the same) through (IV.8) coincide with the conventional operator $\hat{L} = \hat{q} \times \hat{p}$, the operator of the square of the angular momentum determined by (VI.2) **does not** coincide with the square of the operator of the angular momentum. It is not difficult to see that, according to (VI.2), corresponding to the phase-space function $L^2(q, p) = (q \times p)^2$ is the operator

$$(VI.3\,a) \qquad (\hat{L})^2 - 2i\hbar\hat{q}.\hat{p} \equiv (\hat{L})^2 - 2\hbar^2 r \frac{\partial}{\partial r},$$

while in the conventional interpretation, to the square of the angular momentum corresponds the operator

$$(VI.3\,b) \qquad (\hat{L})^2 = (\hat{q} \times \hat{p})^2.$$

This means that in the approach offered the operator of the square of the angular momentum **does not** coincide with the square of the operator of the angular momentum. It must be remembered, however, that the operators do not play a fundamental role in the approach offered, so that the incongruence indicated is actually of no significance.

A second contradiction, which is essential on a conceptual plane, between the conventional approach and the one proposed, is the contradiction on the problem of the operator of the radial momentum.

It is usually said that the momentum operator is vectorial and (in q-representation) has the form

$$(VI.4) \qquad \hat{p} = -i\hbar\mathrm{grad}.$$

According to this, in spherical coordinates we must have

$$(VI.5\,a) \qquad \hat{p}_r = -i\hbar\frac{\partial}{\partial r}.$$

Precisely this operator (i.e. $\hat{p}_r$) corresponds to the dynamic characteristics radial momentum (i.e. p_r) by virtue of the correspondence rule (VI.2). Consequently,

in the formulation proposed the operator of the radial momentum is the operator (VI.5a).

However, this operator is unacceptable for the conventional interprtation since it is not Hermitian. That is why, as "genuine" operator of the radial momentum is taken [7, 22, 23] the operator

$$(VI.5b) \qquad -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \equiv \hat{p}_r - i\hbar\frac{1}{r},$$

which obviously is not the radial component of the momentum operator, but it is Hermitian (at least formally — [23]–Ch.7, Sec.8). Of course, there is also difference in the determination of the squares of these quantities. Whereas in the approach offered the opepator which, according to (IV.8), corresponds to the square of the radial momentum is

$$(VI.6a) \qquad -\hbar^2 \frac{\partial^2}{\partial r^2},$$

the conventional operator of the square of the radial momentum (the square of the (VI.5b) operator) is

$$(VI.6b) \qquad -\hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) \equiv -\hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

The two operators studied — that of the radial momentum and that of the square of the radial momentum — participate as addenda in the operator of the kinetic energy. It is an essential fact that the latter operator *is the same* in both the conventional and in the proposed approach: it turns out that the differences in the components compensate each other. The identity

$$(VI.7) \qquad -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{2m}\frac{1}{r^2}(\hat{L})^2 \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{1}{2m}\frac{1}{r^2}\left((\hat{L})^2 - 2\hbar r\frac{\partial}{\partial r}\right),$$

means precisely that.

The difference between the two approaches is that in the conventional one the summand

$$(VI.8) \qquad -\frac{\hbar^2}{2m}\frac{2}{r}\frac{\partial}{\partial r}$$

is related to the radial part of the operator of the kinetic energy (i.e. to the radial kinetic energy), whereas in the approach offered the same summand is related to the second term of the right-hand side of (VI.7), i.e. to the *transversal* kinetic energy. Thus in the proposed approach the transversal kinetic energy is a sum of two components: the first one connected with the operator

$$(VI.9) \qquad \frac{1}{2m}\frac{1}{r^2}(\hat{L})^2$$

and the second one with the operator (VI.8). In the quanton approach the interpretation of these two components is simple: the energy connected with (VI.8) is due

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to the fact that the quantons in the general case have not only a radial component of the velocity but also an orthogonal component to the radial one, i.e. a transversal component of the velocity. Consequently in general both the momentum and the kinetic energy possess also the two kinds of components — radial and transversal. When the electron cloud revolves as a whole, e.g. hydrogen atom in p, d ..., etc. (though not s!) state, the transversal (drift) kinetic energy has, as its operator, the standard one (VI.9). In the s-state there is no systematic (drift) transversal velocity, the chaotic transversal momenta have zero average value (at each point q), but their squares are **not** compensated and they make a contribution to the kinetic energy through the operator (VI.8). We shall not dwell here on the proofs of these assertions (which are actually trivial).

In the concluding part of this Section we shall remark that, within the framework of the approach proposed, the inequality

(VI.10)
$$(\Delta M_z)^2 (2\pi)^2 \geqq h$$

proves to be universally valid, unlike the conventional approach in which $\Delta M_z = 0 = (\Delta M_z)^2$ in the s-state. This is so, because in the quanton approach the dispersion of the angular momentum is determined by the sum of the operators (VI.9) and (VI.8), and not only by the operator (VI.9). On the other hand, for each function Ψ, such that $r^3 \Psi^* \Psi \xrightarrow[r \to \infty]{} 0$, the following readily verifiable equality holds

(VI.11)
$$\int \Psi^* \left(-2\hbar^2 r \frac{\partial}{\partial r} \right) \Psi \, dV = 3\hbar^2.$$

It involves the second operator from the brackets in the right-hand side of (VI.7) and leads directly to (VI.10).

VII. QUANTONS, HEISENBERG RELATIONS, HIDDEN PARAMETERS

Let us consider a concrete example — say, an electron with mass $m \approx 10^{-27}$ g (an electron in the specific sense). In this case the characteristic frequency is $N = (mc^2/h) \approx 10^{20}$ sec^{-1} and this is the frequency with which the effective quantons of the electron appear in the entire physical space. Consequently it may be said that at least 10^{20} times per second the electron has precisely determined dynamic characteristics (mass $\overset{n}{M}$, momentum $(\overset{n}{M}/m)\overset{n}{p}$, energy $\overset{n}{E}$, etc.) localized in a precisely determined instant $\overset{n}{t}$ of time in a precisely determined point $\overset{n}{q}$ of physical space, although all pairs $(\overset{n}{q}, \overset{n}{p})$ are different. Concerning the position of the electron, it must be pointed out that $\overset{n}{q}$ does **not** mark the position of the electron in the instant $\overset{n}{t}$: the electron which consists not only of the generation of quantons but of the wave function as well, is **not** a point-like object at all. It is an extended object whose volume coincides with the support of $\Psi (\Psi(q) \neq 0 \leftrightarrow q \in \Delta V$, where ΔV is

the electron volume). However, $\overset{n}{q}$ determines the possibility for localization at the instant $\overset{n}{t}$.

Under these conditions the meaning of the Heisenberg relations becomes quite clear: these are connections between the dispersions of the physical characteristics of (the quantons of) the electron, averaged for time $\Delta t > T_m$. The simplest example is the free electron with uniquely determined momentum (de Broglie's flat wave). In this case all quantos of the electron appear with one and the same velocity $v = p/m$ and have one and the same average in time (for the interval T_m) momentum p (consequently $\Delta p = 0$), but the appearances of the quantons in physical space are entirely uncorrelated — each point q being equally probable for the appearance of a quanton — consequently the dispersion of the coordinates of the various quantons is $\Delta q = \infty$. In this sense the free electron differs quite substantially from the classical material point, since the electron spreads in the entire infinite physical space.

Another example of dispersion is that of the angular momentum studied in the preceding Section. In a similar way it is possible to treat also the dispersion of each dynamic characteristic. In each case of this kind, within the framework of the proposed theory, we may speak of indeterminacy (or more accurately non-unique, i.e. polyvalent determinancy) of a given physical characteristic only when there are more than one quanton for the time interval during which the given characteristic is averaged.

Closely related to the problem of dispersion is that of the hidden parameters. According to the well known von Neumann's theorem about the hidden parameters, there are no dispersion-free quantum-mechanical ensembles. Of cource, it is an error to interpret this result as a proof for the non-existence of hidden parameters. Von Neumann's result merely states that if hidden parameters exist their dispersion **cannot** be smaller than the corresponding specific quantum-mechanical dispersion intrinsic to an individual microobject. The theory proposed in the present paper is completely compatible with this result. According to this theory, the individual electron itself represents a statistical ensemble (generation) of quantons.

The dispersions in this ensemble (according to its definition) are the well known quantum-mechanical dispersions (the density F satisfies the marginal conditions (I.2)). It follows that no ensemble of electrons can have dispersions smaller than the dispersions of the individual electron. However, this is valid only if the individual electron is studied at time intervals sufficiently greater than T_m; otherwise the dispersions lose their meaning for the separate electron, but T_m is the time boundary of the statistical quantum-mechanical description of the individual electron. The fluctuation phenomena (separate in time appearance of quantons) begin to dominate completely below that boundary. Concerning fluctuation phenomena due to the separate appearances of quantons in space, one such typical phenomenon is obtained at the fluctuation-type localization of the electron (e.g. at the two slits experiment) in separate microvolumes which are macroscopically perceived as separate points.

VIII. COMPARISONS

The proposed approach has both a lot in common and certain essential differences from the classical mechanics of a material point, from the coventional phase-space formulation of quantum mechanics, and from the conventional (orthodox) quantum mechanics. The similarities lie in the fact that the quantons are nothing but basic functions in the extended phase — (QTP_m) space which is the same both in the classical mechanics of a material point and in quantum mechanics. Let us discuss this problem.

Quantons and classical mechanics. As we saw, quantons are at hand in classical mechanics as well (cf. II.4 and III.9). Only, in the classical mechanics of a material point their set is assumed to be (actually without any intrinsic necessity) cotinuous (not discrete) and also arranged along a QTP_m-trajectory. The concept proposed in this paper is in a certain sense, a direct generalization of classisal mechanics, since each classical QTP_m-density of a material point may be seen as one (continuous) generation of quantons. It is obvious, however, that there are no intrinsic grounds to believe that each generation of quantons must be arranged on one QTP_m-trajectory. What has been said is sufficient to warrant the assertion that classical mechanics is obviously subject to generalization along the line of the structure of the set of quantons which (though implicitly) has always been present in it.

Quantons and the phase-space formulation of quantum mechanics. Because of what has been presented thus far, the author believes it to be evident that Heisenberg's relations can neither be an obstacle nor do they make problematic the introduction and utilization of phase-space in quantum mechanics. On the other hand, the introduction of the dualism of characteristics — anticharacteristics (i.e. the introduction of the binary characteristics $(A, -A)$), combined with the dualization of the phase-space density (the introduction of the binary densities $F^{\pm}$) liquidates the difficulty involved in the indefiniteness of Wigner's distribution even without the use of the quantons. However, the dualism of the bicharacteristics and bidensities finds its natural grounds in the quantons-antiquantons dualism. This latter dualism is, in itself, "more natural" than a monism of positive quantons only (just as the existence of the set of real numbers may be considered as "more natural" than the existence only of the set of positive numbers).

Besides Heisenberg's relations and the indefiniteness of the Wigner's distribution there is yet another sourse of suspicion toward the phase-face formulation — Cohen's no-go theorem (the violation of (VI.1) for each Wigner distribution [10]. From the viewpoint of the operator theory of observation, this incongruence makes virtually impossible any phase-space formulation of quantum mechanics. From the quanton viewpoint this incongruence leads to rejection of the operator theory of observation. As a matter of principle, this problem can be solved by experiment. For instance, if it is experimentally established that Heisenberg's relation (VI.10) is valid in an s-state as well (e.g. of the hydrogen atom), this would mean experimental refutation of the operator theory of observation since, according to this theory, in the s-state the dispersion of the angular momentum is equal to zero.

Cohen's no-go theorem could be fatal to any phase-space formulation of quantum mechanics if the following proposition turns out to be true:

There exists a set of the following experiments: 1) . . . ., 2) . . . ., ..., n) . . . ., whose results could be explained *only* by the operator theory of observation (i.e. they **cannot** be explained for any selection of Wigner's distribution).

No one has ever shown this proposition to be true. Consequently Cohen's no-go theorem cannot be taken as experimental argument against the phase-space formulation. Nor can it constitute a theoretical argument from the proposed phase-space version, which is not only independent from the operator theory of observation but also claims to establish its origin. Let us remark once again that in the proposed quanton phase-space formulation the theoretical status of the "observables" (i.e. of the dynamic characteristics) is exactly the same as in the classical physics.

IX. CONCLUSION

In conclusion we shall summarize the results provided by the concept of quantons and shall dwell on the problem of the price involved. We shall also say a few words about the trend of development of that concept.

First of all it may be maintained that if the quantons really exist, then their study amounts to studying the ultramicrostructure of matter — an objective sufficiently honourable and promising in itself. In addition, the existence of the quantons, which is directly connected with the quantization of action, would mean that:

1. Quantum mechanics without quantons is essentially incomplete, just as classical mechanics would have been incomplete without them, i.e. without the object of material point. This incompleteness is compensated by the *ad hoc* introduction of the operator theory of observation.

2. Hidden parameters do exist but they are non-local (in accordance with the results obtained by B e l l [24]), stochastic (in accordance with von Neumann's theorem) and ultra-fast variable (which has not been assumed so far). Univalent (Laplace) determinism is not relevant to quantum mechanics, but the principle of sufficient cause continues operating in it as well.

3. The particle-wave dualism finds a rational explanation, together with the problem of the reduction, the relations of indeterminacy, etc., without reference to a special theory of observations. In contrast wiht the orthodox approach, the operators "observables" are defined via the dynamic characteristics, and not vice versa.

4. There are quantities which are, in principle, measurable (there is no *theoretical* ban on their measurement), whose existence contradicts the orthodox interpretation: these are the ultramicrodensity (III.19), the local densities $F^{\pm}(\pm A) \equiv FA$ (cf. III. 21), the recording time (V.3), and the dispersion (VI.10). The possibility for the practical realization of the corresponding experiments is probably of decisive psychological significance, but it is of no significance to the logic of the theory.

The price which must be paid for the discretization of the set of the quantons is to give up the continuity of the QTP_m-*trajectory* of the classisal particle and to

replace it with the discrete random QTP_m-*lattice* of the quantons. It is natural for such a step to be a psychological barrier, but this is a matter of psychology and not of logic. From a logical point of view, an atomism which goes as far as discretization in time (this being precisely the underlying idea of the quanton approach) seems undoubtedly more consistent than the classical atomism in which discreteness holds only in space, and not in time. Let us nothe once again that the discreteness in time which we have in mind is in fact discreteness (quantization) of action (cf. III. 26).

Another psychological barrier may be the widespread though unfounded belief that causally connected events may form only timelike pairs in Minkowski space. The quanton approach is in obvious contradiction to that requirement. For instance, in the case of a flat wave — free de Broglie "particle", the distance $|\overset{n+1}{q} - \overset{n}{q}|$ is by necessity unlimited, while $\overset{n+1}{t} - \overset{n}{t} = T_m = h/mc^2$ always. The rejection of "relativistics" causality makes it necessary to treat the problem of the essence of the (special) theory of relativity.

The author of this article shares the view that the essence of the special theory of relativity lies in establishing a mathematical model of the realities that are perceived intuitively as space, time and matter. To wit: the space-time has as its mathematical model the well-known (four dimensional) Minkowski space; matter is modelled by likewise well-known conservative energy-momentum tensor. The structure of this tensor is the object of a series of additional postulates, among which the postulate about the "relativistic" causality may be included as an *independent* postulate, although (without any fears) it may also be left out precisely because it is an independent postulate and not a deduced corollary. Consequently, rejection of the "relativistic" causality is logically admissible. Causality in our approach is ensured by the requirement for the existence of a *privileged arrangement* in the set of the quantons making up a generation (in a relativistic formulation, the time referred to in (III.12) is the proper time of the electron).

The problems arising from the relativistic aspects of the quanton approach are treated by the present author in [25].

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САМОСТОЯТЕЛЬНАЯ ФАЗОВО-ПРОСТРАНСТВЕННАЯ ФОРМУЛИРОВКА КВАНТОВОЙ МЕХАНИКИ КАК СТАТИСТИКА ВИРТУАЛЬНЫХ ЧАСТИЦ

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(Р е з ю м е)

Предлагается подход к квантовой механике, основаный на следующих положениях: 1. конвенциональное квантово-механическое описание физической реальности неполько; 2. причинные влияния, распространяющиеся бистрее света, волне совместимы с теорией относительности.

В дополнение к стандартному квантовомеханическому формализму мы вводим новый объект, который до сих пор не рассматривался. Этот объект является

упорядоченным множеством виртуальных частиц, которы мы называем "генерация квантонов".

Генерация квантонов является объектом достаточно различающимся от классической материальной точки, чтобы обладать полностью квантово-механическим поведением и квантово-механическими свойствами и в то же время в достаточной степени подобен классической материальной точке, чтобы описать его как реальный объект в классическом фазовом пространстве. В принципе существование квантонов может быть установлено экспериментальным образом.